PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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A Note on Super-Replicating Strategies

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Phil. Trans. R. Soc. Lond. A 1994 **347**, 485-494
doi: 10.1098/rsta.1994.0058

A note on super-replicating strategies

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The standard Black–Scholes option pricing methodology fails in the presence of transaction costs because portfolios that exactly replicate the option pay-off no longer exist. Several alternative approaches have been proposed; our purpose is to examine one of them which is based on the idea of 'super-replicating' portfolios. It is argued that this approach does not lead to a viable theory of option pricing in continuous time.

1. Introduction

The Black–Scholes theory of option pricing relies on the existence of replicating portfolios: given a certain initial endowment, an investor can form a dynamic portfolio in a risky stock and a risk-free bank whose value at some later time T exactly coincides with that of, for example, a European call option with exercise time T and some given strike price. An arbitrage opportunity exists for one party or the other if any other price than this initial endowment – the Black–Scholes price – is paid for the option at time 0. These matters are discussed in many textbooks, for example Cox & Rubinstein (1985) or Merton (1992). The argument fails when there are transaction costs or other forms of market friction, for then, replicating portfolios generally no longer exist. A variety of alternative approaches has been proposed to deal with this, the first important work being due to Leland (1985) who examined the effects of periodic rebalancing of a hedging portfolio. The purpose of the present paper is to take a critical look at another approach based on the idea of super-replicating strategies (SRS). An SRS for an option is a trading strategy such that the value of the corresponding portfolio is at least as great as that of the option at the exercise time. This idea is in fact used in the standard Black–Scholes framework in connection with pricing American options; see Karatzas (1989). In the discrete-time binomial model, perfect replication is still possible even in the face of transaction costs, but Bensaid *et al*. (1992) uncovered the intriguing fact that super-replication may be cheaper, providing therefore a tighter bound on feasible option prices.

We believe, however, that super-replicating strategies do not form the basis for a viable theory of option pricing in continuous time. For a call option, there is always one trivial SRS, namely to buy and hold one share, the corresponding initial endowment being of course the share price at time 0 plus any transaction costs on the purchase. We conjecture that this is the cheapest SRS. Although no proof of this is as yet available, there is strong evidence in favour of it. If true, option pricing based on SRS should be discarded in favour of alternative approaches based on utility maximization (see Hodges & Neuberger 1989 or Davis *et al*. 1993), which 'soften' the penalty for under-replication.

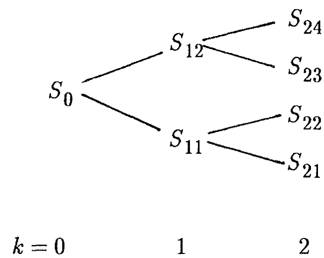

Figure 1.

The paper proceeds as follows. In §2 the discrete-time binomial model is described and the existence of replicating and possibly cheaper super-replicating strategies demonstrated. The continuous-time model including transaction costs is formulated in §3, following Davis & Norman (1990) and Davis *et al.* (1993). In the final section (§4) our conjecture is stated formally and the evidence for it presented.

2. Discrete-time binomial models

Option pricing based on a binomial stock price model was introduced by Cox *et al.* (1975) and is discussed in §10.2 of Merton (1992). Let us consider a 2-period model with time instants $k = 0$, 1, 2 and a single stock whose price evolves as shown in figure 1.

The initial price is $S(0) = S_0$ and this moves to $S(1) = S_{11}$ or S_{12} at time $k = 1$, etc. It is not necessary to specify the probabilities of these transitions: the only requirement is that every path in the tree must have strictly positive probability. By convention, $S_{11} < S_{12}$, $S_{21} < S_{22}$, $S_{23} < S_{24}$. There is also a riskless asset, the 'bank', which yields a return R per dollar over one period. To rule out arbitrage we clearly must have $S_{11} < RS_0 < S_{12}$, $S_{23} < RS_{12} < S_{24,}$, $S_{21} < RS_{11} < S_{22}$ as otherwise riskless profits can be made by borrowing from bank to invest in stock or vice versa.

Suppose that a contingent claim has value $h(S(2))$ at time $k = 2$. It is easy to see how to form a replicating portfolio for this. Let N_{11}, B_{11} [N_{12}, B_{12}] denote the number of stock units and the amount invested in bank at time 1 when the stock price is $S(1) = S_{11}[= S_{12}]$. Then for perfect replication we must have

$$\left.\begin{aligned} RB_{11}+N_{11}S_{21} &= h(S_{21}), \quad & RB_{12}+N_{12}S_{23} &= h(S_{23}), \\ RB_{11}+N_{11}S_{22} &= h(S_{22}), \quad & RB_{12}+N_{12}S_{24} &= h(S_{24}). \end{aligned}\right\} \tag{2.1}$$

These equations are easily solved to give

$$\left.\begin{aligned} N_{11} &= [h(S_{21})-h(S_{22})]/(S_{21}-S_{22}), \\ N_{12} &= [h(S_{23})-h(S_{24})]/(S_{23}-S_{24}), \\ B_{11} &= [S_{21}H(S_{22})-S_{22}H(S_{21})]/R(S_{21}-S_{22}), \\ B_{12} &= [S_{23}H(S_{24})-S_{24}H(S_{23})]/R(S_{23}-S_{24}). \end{aligned}\right\} \tag{2.2}$$

The value of the hedging portfolio at time 1 is then $F_{11} = B_{11}+N_{11}S_{11}$, if $S(1) = S_{11}$, and $F_{12} = B_{12}+N_{12}S_{12}+N_{12}S_{12}$, if $S(1) = S_{12}$. We can now perform a similar analysis to determine N_0 and B_0, the number of stock units and dollars held in bond at time 0 so that F_{11} and F_{12} are perfectly replicated at time 1. The price of the contingent claim h at time 0 is then $F_0 = B_0+N_0S_0$, the value of the replicating portfolio.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

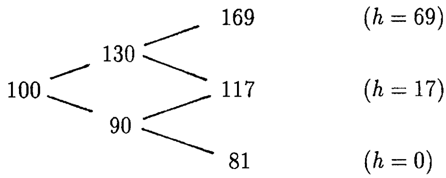

Figure 2.

As an example, take $R = 1$ and consider the values shown in figure 2. (At each stage the price moves from S to uS or dS where $u = 1.3$, $d = 0.9$.)

The contingent claim h is an at-the-money call option, so that $h(S(2)) = [S/2) - 100]^+$. (Notation: $[X]^+ = \max\{X, 0\}$.) We find that the option value at time 0 is $F_0 = 10.69$ and the initial composition of the replicating portfolio is $N_0 = 0.644$, $B_0 = -53.71$. The option value F_0 can be expressed as the discounted expectation with respect to a risk-neutral probability distribution, but we do not need this result here.

Perhaps surprisingly, perfect replication is still possible when transaction costs are introduced. This is covered in §14.2 of Merton (1992) and in Boyle & Vorst (1992) and Edirisinghe *et al.* (1993). Let us maintain the 2-period model of figure 1, but assume that a fraction λ of the amount transacted is paid on all movements in and out of stock and that settlement at time $k = 2$ is in cash. The latter is just a convention and the argument below works *mutatis mutandis* for other forms of settlement, for example delivery of one stock unit in return for a cash payment of K for an in-the-money call option with strike price K. Let N_{ij}^{λ}, B_{ij}^{λ} denote the number of units of stock and the dollars held in bond for the new model, so that $N_{ij} = N_{ij}^{0}$ etc. where N_{ij} is the quantity appearing in the original transaction-cost-free model. The requirements (2.1) for exact replication at time $k = 2$ now become

$$\begin{aligned} RB_{11}^{\lambda} + (1-\lambda) N_{11}^{\lambda} S_{21} &= h(S_{21}), \quad & RB_{12} + (1-\lambda) N_{12}^{\lambda} S_{23} &= h(S_{23}), \\ RB_{12}^{\lambda} + (1-\lambda) N_{11}^{\lambda} S_{22} &= h(S_{22}), \quad & RB_{12} + (1-\lambda) N_{12}^{\lambda} S_{24} &= h(S_{24}), \end{aligned}$$

as the portfolio must be cashed out at time 2, giving

$$\begin{aligned} N_{11}^{\lambda} &= (1/(1-\lambda)) N_{11}, \quad & N_{12}^{\lambda} &= (1/(1-\lambda)) N_{12}, \\ B_{11}^{\lambda} &= B_{11}, & B_{12}^{\lambda} &= B_{12}. \end{aligned}$$

Because $S_{11} < S_0 < S_{12}$ it is reasonable and, as it turns out, correct to suppose that $N_{11}^{\lambda} < N_0^{\lambda} < N_{12}^{\lambda}$. Thus the portfolio value required before rebalancing at time $k = 1$ is, when $S(1) = S_{11}$,

$$\begin{aligned} F_{11}^{\lambda} &= S_{11} N_{11}^{\lambda} + B_{11}^{\lambda} + \lambda (N_0^{\lambda} - N_{11}^{\lambda}) S_{11} \\ &= (1-\lambda) S_{11} N_{11}^{\lambda} + B_{11}^{\lambda} + \lambda N_0^{\lambda} S_{11} \\ &= F_{11} + \lambda N_{11}^{\lambda} S_{11}. \end{aligned}$$

When $S(1) = S_{12}$ the corresponding amount is

$$\begin{aligned} F_{12}^{\lambda} &= S_{12} N_{12}^{\lambda} + B_{12}^{\lambda} + \lambda (N_{12}^{\lambda} - N_0^{\lambda}) S_{12} \\ &= F_{12} + \lambda (2N_{12}^{\lambda} - N_0^{\lambda}) S_{12}. \end{aligned}$$

To provide these values, we must have

$$\begin{aligned} RB_0^{\lambda} + N_0^{\lambda} S_{11} &= F_{11} + \lambda N_0^{\lambda} S_{11}, \\ RB_0^{\lambda} + N_0^{\lambda} S_{12} &= F_{12} + \lambda (2N_{12}^{\lambda} - N_0^{\lambda}) S_{12}, \end{aligned}$$

giving

$$N_0^\lambda = (F_{12} - F_{11} + 2\lambda N_{12}^\lambda S_{12})/[(1+\lambda)S_{12} - (1-\lambda)S_{11}],$$
$$B_0^\lambda = [(1+\lambda)F_{11}S_{12} - (1-\lambda)F_{12} - 2\lambda(1-\lambda)N_{12}^\lambda S_{12}S_{11}]/[R(1+\lambda)S_{12} - (1-\lambda)S_{11})].$$

The cash required to set up this portfolio is then

$$F_0^\lambda = B_0^\lambda + (1+\lambda)N_0^\lambda S_0.$$

Thus F_0^λ is an upper bound for the price a buyer would be prepared to pay for the contingent claim, since at any higher price he or she is certainly providing the writer with a riskless profit.

Let us take (just for illustrative purposes) $\lambda = 0.2$ in the example of figure 2. Then we find that $N_0^\lambda = 1.080$ and $B_0^\lambda = -30.68$. Thus the initial value of the replicating portfolio is 77.32 ignoring the set-up cost, or $F_0^\lambda = 98.92$ including the set-up cost. However, as pointed out by Bensaid *et al.* (1992), there are strategies that are clearly better than this. For example, suppose we take $N_0 = N_0^\lambda = 1.08$, $B_0 = -77$ and do no trading at time $k = 1$. Then the cashed-out values of the portfolio at $k = 2$ are 69.02 and 24.01, when $S(2) = 169$, 117 respectively, whereas the 'book' value of the portfolio is 10.48 when $S(2) = 81$. Because, in the latter case, nothing has to be delivered, there is no reason to cash out the portfolio, and the position is therefore more than covered at all values of $S(2)$, at an initial cost of 52.6 (or 31 without the set-up costs), far less than the cost of the replicating portfolio.

One of the main uses of the binomial formulation of option pricing theory is as an approximation to the continuous-time case in which, as outlined in §3 below, the stock price is modelled as a geometric brownian motion with volatility (or standard deviation) parameter σ. This is equivalent to approximating the logarithm of the stock price by a discrete random walk. We take an n-period model in which $S(k+1)$ is equal to $uS(k)$ or $dS(k)$ where $d < 1 < u$. If in the continuous time model the trading interval is $[0, T]$ and the riskless interest rate r then the appropriate scaling is obtained by taking $h = T/n$, $u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$ and $R = e^{rh}$. This approximation is described very clearly in Cox & Rubinstein (1985). In an ingenious analysis, Boyle & Vorst (1992, Theorem 3) show that if one applies this scaling to the binomial model with transaction costs, then for large n and small λ the initial value of the replicating portfolio for a call option is approximately equal to the Black–Scholes value but with a modified variance given by

$$\sigma_n^2 = \sigma^2[1 + (2\lambda/\sigma)\sqrt{(n/T)}].$$

Recall that the 'delta' (i.e. the number of stock units) in the Black–Scholes replicating portfolio is $\Delta(T, S) = N(d_1)$ where N is the cumulative normal distribution function and

$$d_1 = [\ln(S/K) + (r + \tfrac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{(T-t)}.$$

Thus $\Delta(T, S) \to 1$ as $\sigma \to \infty$ and Boyle & Vorst's results show that as the number of periods increases the initial position in the stock converges to one while the position in bond converges to zero. Hence the replicating portfolio is for most periods close to a buy-and-hold policy of simply purchasing a unit of stock at time 0. Such a policy clearly covers the option position at time T.

Based on the above considerations, Bensaid *et al.* (1992) analyse hedging problems with transaction costs, by considering super-replicating strategies (SRS). Suppose $h(S_T)$ is the value of a European contingent claim at exercise time T, S_T being the stock price at that time. An SRS is a dynamic portfolio whose value X_T at time T satisfies $X_T \geqslant h(S_T)$ with probability one. We have already seen that – at first sight paradoxically – the cheapest SRS may be cheaper than the initial cost of the replicating portfolio if transactions costs are present. Thus the SRS concept gives tighter bounds on option values. Bensaid *et al.* (1992) give a constructive procedure for calculating the cheapest SRS in the binomial model. They do not however consider asymptotics, i.e. what happens when the model is rescaled and the number of periods increases to infinity as described above. Rather than do this, we will, in the following sections, examine SRS directly in a continuous-time setting.

3. A continuous-time model with transaction costs

Our continuous-time model, which is similar to the one used by Davis & Norman (1990), is a slight variation of the standard geometric brownian motion model. Let (w_t) be a brownian motion with natural filtration $(\mathscr{F}_t)$, defined on some probability space $(\Omega, \mathscr{F}, P)$. The market contains a single risky asset ('stock') whose price (S_t) satisfies

$$\mathrm{d}S_t = \alpha S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}w_t, \quad S_0 > 0.$$

Here α, $\sigma > 0$ are given constants. There is also a riskless asset ('bank') paying a constant interest rate r. Let (y_t) and (z_t) be the processes of dollar holdings in the stock and bank respectively. A trading strategy is a pair (L_t, U_t) of $(\mathscr{F}_t)$-adapted, right-continuous increasing processes, and the evolution of portfolio holdings (y_t, z_t) in response to such a trading strategy is defined by the following equations:

$$\mathrm{d}y_t = \alpha y_t \,\mathrm{d}t + \sigma y_t \,\mathrm{d}w_t + \mathrm{d}L_t - \mathrm{d}U_t, \qquad y_0 = y,$$
$$\mathrm{d}z_t = r z_t \,\mathrm{d}t - (1+\lambda)\,\mathrm{d}L_t + (1-\lambda)\,\mathrm{d}U_t, \quad z_0 = z,$$

where (y, z) is the initial portfolio composition and $\lambda \geqslant 0$ is the transaction cost parameter. Note that purchase of \$dL of stock requires a payment of \$$(1+\lambda)\,\mathrm{d}L$ from bank, while sale of \$dU of stock realizes only \$$(1-\lambda)\,\mathrm{d}U$ in cash. Thus λ is the fractional transaction cost and all transaction costs are paid from bank. (Note that in the model of Davis *et al.* (1993), (y_t) represents the number of stock units rather than their dollar value.)

To simplify the notation in the following discussion we will, without essential loss of generality, take $r = \alpha = 0$ and $\sigma = 1$. Taking $r = 0$ and $\sigma = 1$ just amounts to a specific choice of numeraire and monetary units, whereas $\alpha = 0$ can always be achieved by replacing P by an equivalent (risk neutral) probability measure. Most of the following argument is concerned with 'almost sure' properties, which are invariant under mutually absolutely continuous substitution of measures. To summarize, our market model is

Stock: $\quad \mathrm{d}S_t = S_t \,\mathrm{d}w_t, \qquad (3.1)$

\$ holdings in stock: $\quad \mathrm{d}y_t = y_t \,\mathrm{d}w_t + \mathrm{d}L_t - \mathrm{d}U_t, \quad y_0 = y, \qquad (3.2)$

\$ holdings in bank: $\quad \mathrm{d}z_t = -(1+\lambda)\,\mathrm{d}L_t + (1-\lambda)\,\mathrm{d}U_t = \mathrm{d}A_t - \lambda \mathrm{d}\breve{A}_t, \quad z_0 = z. \qquad (3.3)$

In (3.3), $A_t := U_t - L_t$ is the net sale of stock and $\breve{A}_t := U_t + L_t$. It is clearly pointless to buy and sell stock at the same time, so we may and shall restrict ourselves to trading strategies such that $(\breve{A}_t)$ is the total variation of the bounded variation transaction process (A_t).

We now describe, following Davis & Norman (1990) or Davis *et al.* (1993), the sort of trading strategies that are optimal from the point of view of maximizing utilities or other functions of the vector of portfolio holdings. Thus, suppose $T > 0$ is a fixed time and we wish to maximize $\mathbb{E}\Phi(s_T, y_T, z_T)$ where Φ is some given function and $\mathbb{E}$ denotes expectation with respect to the probability measure P. For example, taking $\Phi(S, y, z) = (1-\lambda)y + z$ corresponds to maximizing the expected cash value of the portfolio at time T, while $\Phi(S, y, z) = (1-\lambda)y + z - [S-K]^+$ corresponds to maximizing this value if a call option with strike price K has to be settled first. Suppose first that (L_t, U_t) are restricted to be absolutely continuous with derivative bounded by κ, i.e. $\mathrm{d}L_t = l_t\,\mathrm{d}t, \mathrm{d}U_t = u_t\,\mathrm{d}t$ and $0 \leqslant l_t, u_t \leqslant \kappa$. Then we have a 'standard' stochastic control problem, and formal dynamic programming arguments (see Fleming & Rishel, 1975) indicate that the value function

$$V(t, S, y, z) = \sup_{(L,\,U)} \mathbb{E}_{t,(S,y,z)}\,\Phi(S_T, y_T, z_T) \tag{3.4}$$

should satisfy the Bellman equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \tfrac{1}{2}y^2\frac{\partial^2 V}{\partial y^2} + Sy\frac{\partial V}{\partial S\,\partial y}$$
$$+ \max_{l,\,u\in[0,\kappa]}\left\{\left[\frac{\partial V}{\partial y} - (1+\lambda)\frac{\partial V}{\partial z}\right]l + \left[-\frac{\partial V}{\partial y} + (1-\lambda)\frac{\partial V}{\partial z}\right]u\right\} = 0 \tag{3.5}$$

with boundary condition

$$V(T, S, y, z) = \Phi(S, y, z). \tag{3.6}$$

The key point in (3.5) is that the maxima are achieved at the extreme values $\{0, \kappa\}$, the values taken depending on the signs of the terms multiplying l, u in (3.5). Specifically the (t, S, y, z) space splits into three regions:

$\mathscr{B}$ (buy at maximum rate, $l = \kappa, u = 0$): $\partial V/\partial y \geqslant (1+\lambda)\,\partial V/\partial z$,

$\mathscr{S}$ (sell at maximum rate, $l = 0, u = \kappa$): $\partial V/\partial y \leqslant (1-\lambda)\,\partial V/\partial z$,

$\mathscr{N}\mathscr{T}$ (no transactions, $l = u = 0$): $(1-\lambda)\,\partial V/\partial z < \partial V/\partial y < (1+\lambda)\,\partial V/\partial z$.

As $\kappa \to \infty$ this split is maintained, but the transactions in $\mathscr{B}$ and $\mathscr{S}$ take place at 'infinite speed', corresponding to an instantaneous transaction to the boundary of the no-transaction region, implying that the above inequalities in $\mathscr{B}$, $\mathscr{S}$ hold with equality throughout these regions. In the limit (3.5) becomes the variational inequality

$$\max\left\{(1-\lambda)\frac{\partial V}{\partial z} - \frac{\partial V}{\partial y}, -(1+\lambda)\frac{\partial V}{\partial z} + \frac{\partial V}{\partial y}, \frac{\partial V}{\partial t} + \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \tfrac{1}{2}y^2\frac{\partial^2 V}{\partial y^2} + yS\frac{\partial V}{\partial S\,\partial y}\right\} = 0, \tag{3.7}$$

with the same boundary condition (3.6) as before, to be solved in a certain 'solvency region' of (t, S, y, z) space; see Davis *et al.* (1993). This equation is 'degenerate' in that the second order term is not uniformly elliptic (there is no 'noise' in the z-component), and it therefore cannot *a priori* be expected to have a classical ($C^{1,2}$) solution. However, under appropriate conditions on Φ, the variational inequality has a unique solution in the viscosity sense (see Fleming & Soner (1992) for this) which coincides with V defined by (3.4). After a possible initial jump to the boundary of the no-transaction region $\mathscr{N}\mathscr{T}$, the optimal transaction processes (L_t, U_t) are the local times of the portfolio process at the boundaries $\partial\mathscr{B}, \partial\mathscr{S}$ respectively. These are the

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

minimal increasing processes needed to keep the portfolio vector within $\mathcal{N}\mathcal{T}$; they are continuous but singular with respect to Lebesgue measure and increase only on the boundaries $\partial\mathcal{B}, \partial\mathcal{S}$. The optimal portfolio is in the interior of $\mathcal{N}\mathcal{T}$, where $dL = dU = 0$, i.e. no transactions take place, almost all of the time.

4. Super-replicating strategies

Let us now consider a European call option on the stock (S_t) with exercise time T and strike price K. For simplicity of exposition we will assume that transactions are free at times $0, T$. Thus the value of the option at exercise is $[S(T)-K]^+$ and an initial cash amount z can be invested to form a portfolio whose initial composition is $(z-z', z')$ for any z'; after this, transactions incur a proportional charge λ as described in §3.

Definition 4.1. A *A super-replicating strategy* (SRS) *is an initial cash endowment z and a trading strategy (L, U) such that*

$$y_T + z_T \geqslant [S_t - K]^+ \quad a.s.,$$

where (y_t, z_t) is the solution of (3.2), (3.3) *with $y_0 = z - z', z_0 = z'$ for some $z' \in \mathbb{R}$.*

There is one trivial SRS, namely to buy and hold one share. This corresponds to taking $z = S_0$, $z' = 0$, $L_t \equiv U_t \equiv 0$ so that $y_T + z_T = S_T \geqslant [S_T - K]^+$. Based on the behaviour of the discrete-time models discussed in §§1 and 2, the fundamental contention of this paper is

Conjecture 4.2. *The trivial buy-and-hold strategy is the cheapest super-replicating strategy when the transaction cost parameter λ is strictly positive.*

A proof of this statement has eluded us so far, but we have found no counter-example and a considerable amount of evidence in favour of it, which is adduced in the remainder of this section.

Because the buy-and-hold strategy has cost S_0, the conjecture amounts to saying that no strategy with $z < S_0$ is super-replicating. In view of the free transaction at $t = 0$ we can take as initial conditions in (3.2), (3.3) $y_{0-} = S_0 - \epsilon, z_{0-} = 0$ for some $\epsilon > 0$ and the conjecture is established if $P[y_T + z_T < [S_T - K]^+] > 0$ for every trading strategy (L_t, U_t) with this initial condition.

Example 4.3. A strategy that 'nearly' works is to take $L_t = \epsilon$ (i.e. $\Delta L_0 = \epsilon$), $U_t = 0$, i.e. we borrow ϵ from bank and buy a share; then $y_t = S_t$ and $z_t = -\epsilon$, and, assuming $\epsilon < K$, $y_T + z_T < [S_T - K]^+$ on the set $S_T < \epsilon$ which of course has positive probability. Insisting on super-replication with probability one is a very stringent condition.

Using the elementary identity $S \wedge K = S - [S-K]^+$ where $S \wedge K = \min\{S, K\}$ we have $y_T + z_T < [S_T - K]^+$ if and only if $\zeta_T > S_T \wedge K$, where ζ_T is the deficit process

$$\zeta_T = S_t - y_t - z_t,$$

the difference between the value of one share and the 'raw' value of the portfolio. Conjecture 4.2 is therefore established if, with $z_{0-} = 0$, $y_{0-} = S - \epsilon$,

$$P[\zeta_T > S_T \wedge K] > 0$$

or, equivalently,

$$\operatorname*{ess\,sup}_{\omega \in \Omega} (\zeta_T(\omega) - S_T(\omega) \wedge K) > 0. \tag{4.1}$$

This statement can be recast in terms of expectations in the following way. For a bounded adapted process, ψ_t, let Λ_T be the Girsanov density

$$\Lambda_T(\psi) = \exp\left(\int_0^T \psi_s \, \mathrm{d}w_s - \frac{1}{2}\int_0^T \psi_s^2 \, \mathrm{d}s\right).$$

The following lemma is easily established.

Lemma 4.4. *For any integrable $\mathscr{F}_T$-measurable random variable X we have*

$$\operatorname*{ess\,sup}_{\omega\in\Omega} X(\omega) = \sup_{\psi \text{ bounded}} E[\Lambda_T(\psi)X].$$

We now calculate $E[L(\psi)X]$ when X is given by the expression at (4.1).

Theorem 4.5. *Consider an arbitrary continuous trading strategy (L, U) with initial endowment $z_{0-} = 0$, $y_{0-} = S-\epsilon$. Then for any bounded measurable process (ψ_t),*

$$\mathbb{E}\{\Lambda_T(\zeta_T - S_T \wedge K)\} = \epsilon - g(0, S_0) + \tfrac{1}{2}\mathbb{E}\int_0^T \Lambda\psi(\phi - g'S)\,\mathrm{d}t + \lambda\,\mathbb{E}\int_0^T \Lambda\,\mathrm{d}\breve{A}_t. \qquad (4.2)$$

In this equation $\Lambda_t = \Lambda_t(\psi)$, $\phi_t = S_t - y_t$ and $g' = (\partial g/\partial S)(t, S_t)$ where g is the unique solution of the partial differential equation

$$\partial g/\partial t + \tfrac{1}{2}S^2\,\partial^2 g/\partial S^2 = 0, \quad S > 0, \quad t\in[0,T], \quad g(T,S) = S \wedge K.$$

Proof. By standard arguments, $g(t,S) = \mathbb{E}_{t,S}[S_T \wedge K]$. Thus $g(t,s) = S - p(t,S)$, where $p(t,S)$ is the Black–Scholes price of the call option. From (3.1)–(3.3), $\phi_t = S_t - y_t$ satisfies

$$\mathrm{d}\phi_t = \phi_t\,\mathrm{d}w_t + \mathrm{d}A_t, \quad \phi_{0-} = \epsilon, \qquad (4.3)$$

whereas $z_t = A_t - \lambda\breve{A}_t$, so that $\zeta_t = S_t - y_t - z_t = \phi_t - z_t$ satisfies

$$\mathrm{d}\zeta_t = \phi_t\,\mathrm{d}w_t + \lambda\,\mathrm{d}\breve{A}_t, \quad \zeta_{0-} = \epsilon. \qquad (4.4)$$

Also the Girsanov density $\Lambda_t = \Lambda_t(\psi)$ satisfies

$$\mathrm{d}\Lambda_t = Lv\,\mathrm{d}w, \quad \Lambda_0 = 1, \qquad (4.5)$$

whereas from the Itô formula and (4.3) we have

$$\mathrm{d}g(t,S_t) = g'S\,\mathrm{d}w. \qquad (4.6)$$

As A_t is assumed to be continuous we have, by applying the Itô formula to (4.4)–(4.6),

$$\mathrm{d}(\Lambda_t\zeta_t) = \lambda(\phi\,\mathrm{d}w + \lambda\,\mathrm{d}\breve{A}) + \zeta\Lambda\psi\,\mathrm{d}w + \tfrac{1}{2}\Lambda\psi\phi\,\mathrm{d}t,$$

$$\mathrm{d}(\Lambda_t g(t,S_t)) = \Lambda g'S\,\mathrm{d}w + gL\psi\,\mathrm{d}w + \tfrac{1}{2}L\psi g'S\,\mathrm{d}t,$$

and hence

$$\Lambda_T(\zeta_T - S_T \wedge K) = \Lambda_T(\zeta_T - g(T, S_T))$$

$$= \epsilon - g(0,S_0) + \frac{1}{2}\int_0^T \Lambda\psi(\phi - g'S)\,\mathrm{d}t + \int_0^T L(\zeta\psi\phi - g'S - gv)\,\mathrm{d}w$$

The last term is a martingale when ψ is bounded, and (4.2) follows. □

Define $\alpha_t := \phi_t - g'(t, S_t) S_t$. Then Theorem 4.4 states that super-replication fails if and only if there is some process (ψ_t) such that

$$\mathbb{E}\int_0^T \Lambda_t \psi_t \alpha_t \, dt + 2\lambda \mathbb{E}\int_0^T \Lambda \, d\check{A}_t > 2(g(0, S_0) - \epsilon). \tag{4.7}$$

The process (α_t) has an interesting interpretation. Since $g = S - p$ and $\phi = S - y$,

$$\alpha_t = p'(t, S_t) S_t - y_t.$$

Thus α_t is the difference between the dollar value of stock holdings in the Black–Scholes replicating portfolio and in the portfolio corresponding to (L, U). In fact, it is impossible to 'track' the Black–Scholes portfolio exactly with bounded variation strategies, as we now show.

Proposition 4.6. *There is no trading strategy* (L, U) *such that* $\alpha_t \equiv 0$.

Proof. We find, using the Itô formula and (4.3), that

$$d(g' S_t) = -\tfrac{1}{2} S^2 g'' \, dt + (g' S + g'' S^2) \, dw, \tag{4.8}$$

whereas

$$d\phi_t = dA_t + \phi \, dw. \tag{4.9}$$

If $\phi_t \equiv g' S_t$ then as A has bounded variation it must be the case that $dA_t = -(\tfrac{1}{2} S^2 g'') \, dt$ and $\phi = (g' S + g'' S^2)$. But since $\phi = g' S$ the latter equality holds only if $g''(t, S_t) = 0$, whereas from the Black–Scholes formula we know that $g''(t, S) \neq 0$ for any (t, S). □

Just to check (4.7), note that if $\lambda = 0$ and we drop the requirement that A has bounded variation then a strategy A giving $\alpha_t = 0$ can be obtained simply by equating the right-hand sides of (4.8) and (4.9). For this strategy (4.2) implies that super-replication fails if and only if $g(0, S_0) - \epsilon < 0$ or, equivalently, $S_0 - \epsilon < p(0, S_0)$, i.e. if and only if the initial value of the portfolio is less than the Black–Scholes price.

On the positive side, there is one case that is easily disposed of.

Proposition 4.7. *Let* $A = (L, U)$ *be a trading strategy with initial portfolio as in Theorem* 4.4, *and define* α_t *by* (4.8). *Suppose that*

$$\int_0^T \frac{1}{\alpha_t^2} \, dt < \infty \quad a.s.$$

and that $\mathbb{E}\Lambda_T(\psi) = 1$ *when* $\psi_t = 1/\alpha_t$. *Then* (L, U) *is not a super-replicating strategy.*

Proof. Define $\psi_t = a/\alpha_t$ for some $a > 0$. Then $\mathbb{E}\Lambda_T(\psi) = 1$ under the stated conditions and

$$\mathbb{E}\int_0^T \Lambda_t \psi_t \alpha_t \, dt = \mathbb{E}\Lambda_T \int_0^T \psi_t \alpha_t \, dt = aT.$$

Thus inequality (4.7) is satisfied if a is sufficiently large. □

Corollary 4.8. *Super-replication fails if there is some* $\delta > 0$ *such that* $|\alpha_t| > \delta$ *for almost all* (t, ω).

Bearing in mind the description of optimal strategies in §3, the above propositions indicate that the only possible candidates for super-replicating strategies are those that track the Black–Scholes portfolio closely (i.e. keep $|\alpha_t|$ small) by the introduction of suitable reflecting barriers. On the other hand, if the sell and buy barriers are too close together then the transaction cost term in (4.7) will become large and super-replication will fail. To get some hints about the latter phenomenon, consider reflecting brownian motion $x_t = w_t + \xi_t^1 - \xi_t^2$ in an interval $[-b, b]$, where ξ_t^1, ξ_t^2 are the local times at $-b, b$ respectively. An easy argument shows that for $\delta > 0$,

$$\mathbb{E}_x \int_0^\infty \mathrm{e}^{-\delta t}\,(\mathrm{d}\xi_t^1 + \mathrm{d}\xi_t^2) = \frac{\cosh\sqrt{\delta}x}{\sqrt{\delta}\sinh\sqrt{\delta}b} =: v_\delta(x).$$

Now $\lim_{\delta\downarrow 0}\delta v_\delta(x) = 1/b$, showing by the 'final value theorem' of Laplace transforms that for small b,

$$\mathbb{E}_x(\xi_T^1 + \xi_T^2) \approx T/b.$$

Thus certainly the transaction cost term in (4.7) 'blows up' as tighter tracking of the Black–Scholes portfolio is attempted. It is a delicate question whether barriers can be devised that are wide enough apart to keep the second term in (4.7) small while being close enough together so that super-replication is not ruled out by Proposition 4.6. We think not.

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